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Bifurcation analysis on a class of three-dimensional quadratic systems with twelve limit cycles

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ABSTRACT

This paper concerns bifurcation of limit cycles in a class of 3-dimensional quadratic systems with a special type of symmetry. Normal form theory is applied to prove that at least 12 limit cycles exist with 6–6 distribution in the vicinity of two singular points, yielding a new lower bound on the number of limit cycles in 3-dimensional quadratic systems. A set of center conditions and isochronous center conditions are obtained for such systems. Moreover, some simulations are performed to support the theoretical results.

1. Introduction

Since Poincaré [1] initiated the study of limit cycles, many results have been obtained which are particularly motivated by the well-known Hilbert's 16th problem [2]. This problem is related to the following polynomial vector fields,

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are nth-degree polynomial functions. The second part of Hilbert's 16th problem is to find the upper bound on the number of limit cycles in system (1) and study their distributions. This upper bound is called the Hilbert number, denoted by $H(n)$. The best result for $n = 2$ is $H(2) \geq 4$, and the four limit, with (3,1) distribution, were obtained independently by Shi [3] and Chen and Wang [4] more than 40 years ago. Recently, this result was also proved to be true for near-integrable quadratic systems [5]. However, this problem is even not completely solved for general quadratic systems. For $n = 3$, a lot of results on the low bound of $H(n)$ have been obtained. So far the best result for cubic systems is $H(3) \geq 13$ [6,7], with the distribution of the 13 limit cycles around several singular points. If we consider the local version of the second part of Hilbert's 16 problem, then it becomes an investigation on the number of bifurcating small-amplitude limit cycles around a singular point, denoted by $M(n)$. However, even for the local problem it is still very difficult. Up to today, only generic quadratic systems were completely solved by Bautin in 1952 as $M(2) = 3$ [8]. For $n = 3$, many results have been obtained. It was shown that 9 limit cycles could bifurcate from an elementary focus [9–11]. It has been proved by Yu and Tian [12] 12 small-amplitude limit cycles can exist around a center by perturbing a cubic integrable system.

In real applications, the dimension of a system is often larger than two [13–15]. To analyze bifurcations in such a system, one usually first applies center manifold theory to reduce the system to a 2-dimensional system (e.g., see [13,16]), and

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then applies bifurcation theory to study limit cycles. In general, it is difficult to investigate bifurcation of limit cycles in higher-dimensional ($n \geq 3$) dynamical systems. There have been some results on Hopf bifurcation of 3-dimensional polynomial differential systems. It was shown [17–19] that 3-dimensional quadratic systems can have an infinite number of small amplitude limit cycles bifurcating on an infinite number of invariant algebraic surfaces. However, it is still very difficult to find an upper bound on the number of limit cycles in a 3-dimensional quadratic system restricted to one center manifold or two center manifolds. Wang et al. [20] obtained 5 limit cycles enclosing a singular point in a class of 3-dimensional nonlinear dynamical systems. Tian and Yu [21] considered a simple three-dimensional quadratic system and showed that 7 small limit cycles can exist in around a singular point. Du et al. [22] proved that a class of 3-dimensional quadratic systems can have 10 small limit cycles around two symmetric singular points. Further, Yu and Tian [23] proved that 10 limit cycles can exist in a 3-dimensional quadratic system around single center-type singular point. Recently, Guo et al. [24] found 12 small limit cycles in a class of Z_3-symmetric 3-dimensional quadratic systems.

This paper is concerned with bifurcation of limit cycles in the following 3-dimensional system:

$$\begin{aligned}
\dot{x} &= -a_1 xy,\\
\dot{y} &= \delta y + b_3\Big[-u + \frac{1}{2}(u^2 + x^2)\Big] + b_1 y(1-u) + b_2 y^2,\\
\dot{u} &= c_3(u-u^2) + c_1 y(1-u) + c_2 y^2,
\end{aligned} \tag{2}$$

where a dot represents differentiation with respect to time t, a_1, b_1, b_2, b_3, c_1, c_2 and c_3 are real parameters, satisfying $a_1 > 0$, $b_3 > 0$, $c_3 > 0$ and $|\delta| \ll 1$. It is easy to see that system (2) is invariant under the change $x \to -x$, hence the vector field has the so-called “YOU” symmetry (i.e. the system is symmetric with respect to “y-o-u” plane), with two symmetric singular points at (1,0,1) and $(-1, 0, 1)$.

Main attention of this paper is focused on bifurcation of limit cycles in the 3-dimensional system (2). Normal form theory, with the help of a computer algebraic system, is applied to compute the first 6 focus values associated with Hopf-type singular point (1,0,1) in system (2). Moreover, this singular point is shown to be an order-6 fine focus, implying that at most 6 small limit cycles can exhibit around this point. Further, we prove that proper perturbations can be applied to generate 6 limit cycles, and then, in a total, 12 small limit cycles can bifurcate from the two symmetric singular points. Finally, we derive set of center conditions for such 3-dimensional quadratic systems.

2. Preliminary lemmas

Many mathematical methods in the literature can be used to compute the focus values of planar polynomial systems, for example, the Poincaré Takens method [16], the perturbation approach [25], the singular point value method [26], etc. It is well known that such computations for high-dimensional dynamical systems are more demanding. In this paper, the method of normal forms is applied for computing the focus values, which is suitable for general n-dimension dynamical systems. The readers are referred to [16,27] for general normal form theory. For general n-dimensional differential systems with semisimple singularities, the algorithms and Maple programs have been developed for computing the center manifold and normal form, and can be found in [28]. In this section, we shall present some theorems that are needed for the proof of our main result.

As we all know, the normal form of system (1) associated with a Hopf critical point can be described in the form of

$$\begin{aligned}
\frac{d\rho}{dt} &= \rho(v_0 + v_1\rho^2 + v_2\rho^4 + \cdots + v_j\rho^{2j} + \cdots),\\
\frac{d\theta}{dt} &= \omega_c + \tau_1\rho^2 + \tau_2\rho^4 + \cdots + \tau_j\rho^{2j} + \cdots,
\end{aligned} \tag{3}$$

where ρ and θ represent respectively the amplitude and phase a of motion and the coefficients v_j and τ_j are given in terms of the original system's coefficients. v_j is usually called the jth-order focus value of the origin. The zero-order focus value v_0 can be found from linear analysis. The following theorem gives sufficient conditions on the existence of small-amplitude limit cycles. (The readers can find the proof in [13].)

Lemma 1. *Suppose the focus values v_j's depend on j parameters, expressed as*

$$v_i = v_i(\epsilon_1, \epsilon_2, \ldots, \epsilon_j), \quad i = 0, 1, \ldots, j, \tag{4}$$

which satisfy

$$\begin{aligned}
&v_i(0,\ldots,0) = 0, \quad i = 0,1,\ldots,j-1, \quad v_j(0,\ldots,0) \neq 0,\\
\text{and} \quad &\text{rank}\left[\frac{\partial(v_0, v_1, \ldots, v_{j-1})}{\partial(\epsilon_1, \epsilon_2, \cdots, \epsilon_j)}(0,\ldots,0)\right] = j.
\end{aligned} \tag{5}$$

Then, for any given $\epsilon_0 > 0$, there exist $\epsilon_1, \epsilon_2, \ldots, \epsilon_j$ and $\delta > 0$ with $|\epsilon_i| < \epsilon_0$, $j = 1, 2, \ldots, j$ such that the equation $\frac{d\rho}{dt} = 0$ has exactly j real positive roots (i.e. system (1) has exactly j limit cycles) in a δ-ball with its center at the origin.

It is seen from Lemma 1 than we first need to solve a system of multivariate polynomial (focus value v_j) equations. Suppose that the polynomial ring of the multivariate polynomials given in $x_1, x_2, \ldots, x_r$ with the coefficients in R is denoted by $R[x_1, x_2, \ldots, x_r]$. Then, let

$$p(x_1, x_2, \ldots, x_r) = \sum_{i=0}^{m} p_i(x_1, x_2, \ldots, x_{r-1}) x_r^i,$$
$$q(x_1, x_2, \ldots, x_r) = \sum_{i=0}^{n} q_i(x_1, x_2, \ldots, x_{r-1}) x_r^i \tag{6}$$

be two polynomials in $R[x_1, x_2, \ldots, x_r]$, with positive integers m and n as their degrees in x_r, respectively. Then, the so-called Sylvester matrix of p and q with respect to x_r is given as follows:

$$\mathrm{Syl(p, q, x_r)} = \begin{pmatrix} p_m & p_{m-1} & \cdots & p_0 & & & \\ & p_m & p_{m-1} & \cdots & p_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & p_m & p_{m-1} & \cdots & p_0 \\ q_n & q_{n-1} & \cdots & q_0 & & & \\ & q_n & q_{n-1} & \cdots & q_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & q_n & q_{n-1} & \cdots & q_0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} m \end{matrix} \tag{7}$$

The determinant of the above Sylvester matrix, denoted by $\mathrm{Res}(p, q, x_r)$, is called the resultant of p and q with respect to x_r.

Then, we have the following result.

Lemma 2 [29]**.** *Consider two multivariate polynomials $p(x_1, x_2, \ldots, x_r)$ and $q(x_1, x_2, \ldots, x_r)$ in $R[x_1, x_2, \ldots, x_r]$ given by (6). Let $\mathrm{Res}(p, q, x_r) = h(x_1, \ldots, x_{r-1})$. Then, the following holds.*

1. *If the real vector $< \alpha_1, \alpha_2, \ldots, \alpha_r >\in R^r$ is a common zero of the two equations $p(x_1, x_2, \ldots, x_r) = 0$ and $q(x_1, x_2, \ldots, x_r) = 0$, then $h(\alpha_1, \ldots, \alpha_{r-1}) = 0$.*
2. *Conversely, if $h(\alpha_1, \ldots, \alpha_{r-1}) = 0$, then at least one of the following four conditions is true:*
   (a) $p_m(\alpha_1, \ldots, \alpha_{r-1}) = \cdots = p_0(\alpha_1, \ldots, \alpha_{r-1}) = 0$,
   (b) $q_n(\alpha_1, \ldots, \alpha_{r-1}) = \cdots = q_0(\alpha_1, \ldots, \alpha_{r-1}) = 0$,
   (c) $p_m(\alpha_1, \ldots, \alpha_{r-1}) = q_n(\alpha_1, \ldots, \alpha_{r-1}) = 0$,
   (d) *for some $\alpha_r \in R$, $< \alpha_1, \ldots, \alpha_r >$ is a common zero of both $p(x_1, \ldots, x_r) = 0$ and $q(x_1, \ldots, x_r) = 0$.*

3. 12 limit cycles existing in system (2)

Our main result of this paper is presented in this section. Due to the symmetry in the 3-dimensional quadratic polynomial system (2), focus values associated with the singular point $(-1, 0, 1)$ are exactly the same as that of the singular point (1,0,1) So we only need to calculate focus values for the singular point (1,0,1).

We first introduce the scalings,

$$b_1 \to \sqrt{a_1 b_3}\, B_1, \quad b_2 \to a_1 B_2, \quad c_1 \to a_1 C_1, \quad c_2 \to \frac{a_1^{3/2}}{\sqrt{b_3}} C_2, \quad c_3 \to \sqrt{a_1 b_3}\, C_3,$$
$$\delta \to \sqrt{a_1 b_3}\, \delta, \quad x \to x, \quad y \to \frac{\sqrt{b_3}}{\sqrt{a_1}} y, \quad u \to u, \quad t \to \frac{1}{\sqrt{a_1 b_3}} t, \tag{8}$$

into system (2) to obtain

$$\dot{x} = -xy,$$
$$\dot{y} = \delta y - u + \frac{1}{2}(u^2 + x^2) + B_1 y(1-u) + B_2 y^2,$$
$$\dot{u} = C_3(u - u^2) + C_1 y(1-u) + C_2 y^2. \tag{9}$$

Equivalently, we may simply set $a_1 = b_3 = 1$ in (2) to obtain the dimensionless system (9).

Next, we need to compute the focus values of system (9) to study the limit cycles bifurcating around the Hopf critical point (1,0,1). Thus, we introduce the following transformation,

$$x = y_1 + 1, \quad y = x_1, \quad u = u_1 + 1,$$

into system (9) to obtain the following form,

$$\dot{x}_1 = y_1 + \delta x_1 + \frac{1}{2} y_1^2 + \frac{1}{2} u_1^2 - B_1 x_1 u_1 + B_2 x_1^2,$$

$$\begin{aligned} \dot{y}_1 &= -x_1 - x_1 y_1, \\ \dot{u}_1 &= -C_3 u_1 - C_1 x_1 u_1 + C_2 x_1^2 - C_3 u_1^2. \end{aligned} \tag{10}$$

Obviously, the origin of system (10) is a Hopf-type critical point, which corresponds to the singular point (1,0,1) of the system (2).

Our main result is stated in the following theorem.

Theorem 3. *System (2) can have 12 small limit cycles, around the two symmetric singular points (1,0,1) and $(-1,0,1)$ with 6–6 distribution.*

Proof. First, note $v_0 = \frac{1}{2}\delta$. We set $\delta = 0$, and then apply the method of normal forms and the Maple program in [28] to system (10) to obtain the focus values. In particular, v_1 is given by

$$v_1 = \frac{-B_1 C_2 (3C_3^2 + 8)}{2(C_3^2 + 4)C_3}.$$

Setting $v_1 = 0$ yields $B_1 = 0$, under which we get

$$v_2 = \frac{C_2^2}{3C_3(C_3^2+1)(C_3^2+4)(C_3^2+9)} \times \left(15C_1C_3^4 + 12B_2C_3^4 + 4C_3^2 + 116B_2C_3^2 + 147C_1C_3^2 - 108 + 216B_2 + 216C_1\right).$$

Then, setting $v_2 = 0$ results in

$$B_2 = -\frac{15C_1C_3^4 - 108 + 4C_3^2 + 216C_1 + 147C_1C_3^2}{4(3C_3^4 + 29C_3^2 + 54)},$$

and higher focus values are given by

$$v_3 = \frac{C_2^2 F_1}{2C_3^2(C_3^2+1)(C_3^2+16)(C_3^2+9)(C_3^2+4)^2(3C_3^4+29C_3^2+54)^3},$$

$$v_4 = \frac{C_2^2 F_2}{240C_3^3(C_3^2+25)(C_3^2+16)^2(C_3^2+9)^2(C_3^2+1)^2(C_3^2+4)^3(3C_3^4+29C_3^2+54)^5},$$

$$v_5 = \frac{C_2^2 F_3}{92160C_3^4(C_3^2+36)(C_3^2+25)^2(C_3^2+1)^3(C_3^2+16)^3(C_3^2+9)^3(C_3^2+4)^4(3C_3^4+29C_3^2+54)^7},$$

where F_i's $(i = 1, 2, 3)$ are polynomials in C_1, C_2 and C_3, which can be found in the website: http://www.apmaths.uwo.ca/~pyu.

It is not difficulty to see that setting $C_2 = 0$ yields $v_1 = v_2 = v_3 = \cdots = 0$. Actually, $C_2 = 0$ is a center condition, which will be discussed in Section 4.

Since our purpose is to find maximal number of small-amplitude limit cycles bifurcating from the origin of system (10), we choose the coefficients C_1, C_2, C_3 to solve $F_1 = F_2 = F_3 = 0$. If a solution of $F_1 = F_2 = F_3 = 0$ yields $v_6 \neq 0$, then we can properly perturb the coefficient B_1, B_2, C_1, C_2, C_3 and δ to obtain 6 small-amplitude limit cycles. To show this, eliminating C_2 from the equations $F_1 = F_2 = F_3 = 0$ we obtain a solution $C_2 = C_2(C_1, C_3)$, and two resultant equations:

$$\begin{aligned} F_{12} &= C_3(C_3^2+16)(C_3^2+9)(C_3^2+4)(C_3^2+1)F_{12a}(C_1, C_3), \\ F_{13} &= C_3(C_3^2+16)(C_3^2+9)(C_3^2+4)(C_3^2+1)F_{13a}(C_1, C_3), \end{aligned}$$

where $C_2(C_1, C_3), F_{12a}(C_1, C_3), F_{13a}(C_1, C_3)$ can also be found in the website: http://www.apmaths.uwo.ca/~pyu.

Next, we need to solve $F_{12a}(C_1, C_3) = F_{13a}(C_1, C_3) = 0$ to find the solutions of C_1 and C_3. It can be shown that these two equations have 26 sets of real solutions. However, we found that only 6 sets of them satisfy the conditions $F_1 = F_2 = F_3 = 0$ with $C_2 = C_2(C_1, C_3)$. We choose one of them as follows:

$$\begin{aligned} &C_3 = 2.9729262150\cdots, \quad B_1 = 0, \quad B_2 = -0.1044656162\cdots, \\ &C_1 = 0.1117390817\cdots, \quad C_2 = -0.3366772533\cdots. \end{aligned}$$

under which

$$v_1 = v_2 = v_3 = v_4 = v_5 = 0, \quad v_6 = 0.0003352291\cdots \neq 0.$$

In addition, one can directly verify that the Jacobian evaluated at the critical values equals

$$\det\left[\frac{\partial(v_1, v_2, v_3, v_4, v_5)}{\partial(B_1, B_2, C_1, C_2, C_3)}\right] = 0.8185684627\cdots \times 10^{-11} \neq 0,$$

implying, by Lemma 1, that 5 small-amplitude limit cycles can bifurcate from the center-type singular point (the origin) of system (10). Finally, a further small perturbation on δ gives one more small-amplitude limit cycle. Thus, system (2) can have a total of 12 limit cycles.

The proof of Theorem 3 is complete. □

4. Center conditions

In this section, we derive the center condition of system (10), under which both critical points (1,0,1) and $(-1, 0, 1)$ of system (2) become centers.

It is much more difficult to determine whether an equilibrium associated with a purely imaginary pair is a center in a higher dimensional system than that on a plane. Although people often use Lyapunov constants to find possible center conditions, it is impossible to prove a singular point to be a center by verifying all Lyapunov quantities to be zero. There are several methods such as time-reversibility or integrability which can be applied to identify centers in planar systems, but they are not applicable for dynamical systems with dimension higher than two. Therefore, it is necessary to find a manifold in a closed form. Next, we introduce a method to find a global center manifold for general systems, though the process is simple for our system.

To find the above mentioned global center manifold, one can select candidates from invariant algebraic surfaces. It is not difficulty to verify (see, e.g. [30]) whether a surface, defined by a polynomial equation $F(x_1, y_1, u_1) = 0$, is an invariant surface of the following system,

$$\begin{aligned} \dot{x_1} &= P(x_1, y_1, u_1), \\ \dot{y_1} &= Q(x_1, y_1, u_1), \\ \dot{u_1} &= R(x_1, y_1, u_1), \end{aligned} \tag{11}$$

where the polynomials P, Q and R are assumed to have degree less than or equal to m, if and only if

$$\frac{\partial F}{\partial x_1} P + \frac{\partial F}{\partial y_1} Q + \frac{\partial F}{\partial u_1} R = KF, \tag{12}$$

in which K is a polynomial of degree at most $m-1$. It is easy to see that F is a partial integral of (11). The polynomials F and K are called Darboux polynomial and cofactor of system (11), respectively.

In the following, we define the algebraic partial integral of system (10) passing through the origin, and prove that it defines a global center manifold of system (10).

Theorem 4. *For system $(10)|_{\delta=0}$, when $C_2 = 0$, $\Gamma : u_1 = 0$ is an invariant algebraic surface, which defines a global center manifold and the origin is a center restricted to the manifold (i.e., the critical points (1,0,1) and (-1,0,1) of system $(2)|_{\delta=0}$ are centers when $C_2 = 0$).*

Proof. When $C_2 = 0$, system $(10)|_{\delta=0}$ becomes

$$\begin{aligned} \dot{x}_1 &= y_1 + \frac{1}{2} y_1^2 + \frac{1}{2} u_1^2 - B_1 x_1 u_1 + B_2 x_1^2, \\ \dot{y}_1 &= -x_1 - x_1 y_1, \\ \dot{u}_1 &= -C_3 u_1 - C_1 x_1 u_1 - C_3 u_1^2. \end{aligned} \tag{13}$$

Our strategy of obtaining the global manifold is to find a polynomial Darboux factor F for system (13),

$$F(x_1, y_1, u_1) = \sum_{i=1}^{l} F_i(x_1, y_1, u_1),$$

where the integer $l \geq 1$ and F_i is a homogeneous polynomial of degree i, and $F_l \neq 0$. On the other hand, as mentioned above, the polynomial cofactor $K(x_1, y_1, u_1)$ has degree at most one. Thus, we may assume that

$$K(x_1, y_1, u_1) = c_0 + c_1 x_1 + c_2 y_1 + c_3 u_1. \tag{14}$$

Now, we process a searching from $l = 1$ to prove the existence of such an polynomial $F(x_1, y_1, u_1)$. For $l = 1$, we let $F(x_1, y_1, u_1) = a x_1 + b y_1 + c u_1$. Then, we substitute this F and the cofactor K given above into Eq. (12) and compare the coefficients of like powers to obtain the following algebraic equations,

$$g_j(a, b, c, B_1, B_2, C_1, C_3, c_0, c_1, c_2, c_3) = 0, \quad j = 1, 2, \ldots 9, \tag{15}$$

where

$$\begin{aligned} &g_1 = B_2 a - a c_1, \quad g_2 = -a c_2 - b c_1 - b, \quad g_3 = -B_1 a - C_1 c - a c_3 - c c_1, \\ &g_4 = -a c_0 - b, \quad g_5 = \frac{1}{2} a - c_2 b, \qquad g_6 = -b c_3 - c c_2, \end{aligned}$$

$$g_7 = -bc_0 + a, \quad g_8 = \frac{1}{2}a - cC_3 - c_3c, \quad g_9 = -C_3c - cc_0. \tag{16}$$

Solving the algebraic system (15) yields $a = 0,\ b = 0$ and $c = 1$, which results in the Darboux polynomial $F = u_1$.Thus, the surface $\Gamma : u_1 = 0$ defines an invariant algebraic surface.

Next, we show that the surface Γ is indeed a global center manifold of system (13). To achieve this, we compute the gradient of the function $F = u_1$ at the origin to obtain $\nabla F_{(0,0,0)} = (0, 0, 1)$, which is actually a normal vector of the surface Γ at the origin. On the other hand, it is noted that the tangent space of center manifold [30] at the origin is spanned by the vectors

$$e_1 = (0, 1, 0), \qquad e_2 = (-1, 0, 0),$$

which correspond to the purely pair of eigenvalues of the linearized system of (13) evaluated at the origin. One can verify that

$$\nabla F_{(0,0,0)} \cdot e_1 = 0, \quad \nabla F_{(0,0,0)} \cdot e_2 = 0,$$

which implies that the surface Γ has the same tangent space at the origin. Hence, we have proved that the surface Γ is truly a global center manifold of system (13), and we stop the process of searching F_l for $l \geq 2$.

Finally, we show that the origin of system (13) is a center restricted to the center manifold Γ. With $u_1 = 0$, system (13) is reduced to

$$\dot{x_1} = y_1 + \frac{1}{2}y_1^2 + B_2x_1^2, \qquad \dot{y_1} = -x_1 - x_1y_1. \tag{17}$$

Obviously, system (17) is a revertible system because it is invariant under the change $(x_1, y_1, t) \to (-x_1, y_1, -t)$, and thus the origin of system (17) is a center. Thus, the critical points (1,0,1) and (-1,0,1) of the system (2) are centers when $\delta = 0$ and $C_2 = 0$.

This completes the proof. □

5. Period constants and isochronous center conditions

In this section, we derive a set of isochronous center conditions at the origin of system (10), under which the singular points (1,0,1) and $(-1, 0, 1)$ of system (2) becomes isochronous centers.

Theorem 5. *For system (10), assume $\delta = 0$ and $C_2 = 0$. Then,*

(1) *the origin is a rough center if $B_2 \neq -2$ and $B_2 \neq -\frac{1}{2}$;*
(2) *the origin is an isochronous center if and only if $B_2 = -2$ or $B_2 = -\frac{1}{2}$.*

Proof. When $\delta = 0$ and $C_2 = 0$, we obtain the period constants associated with the origin of system (10) as follows:

$$\begin{aligned} \tau_1 &= -\frac{1}{12}(B_2 + 2)(2B_2 + 1), \\ \tau_2 &= -\frac{5}{1728}(B_2 + 2)^2(2B_2 + 1)^2, \\ \tau_3 &= \frac{1}{414720}(194B_2^2 - 1003B_2 + 194)(B_2 + 2)^2(2B_2 + 1)^2, \\ &\quad\vdots \end{aligned} \tag{18}$$

Obviously, as defined in [31], if $B_2 \neq -2,\ B_2 \neq -\frac{1}{2}$, then $\tau_1 \neq 0$, indicating that the origin of system (10) is a center of order zero (i.e. a rough center).

Next, we consider the isochronous center conditions. When $\delta = 0$ and $C_2 = 0$, restricted to the center manifold Γ, system (10) is reduced to (17). When $B_2 = -2$, (17) becomes

$$\dot{x}_1 = y_1 + \frac{1}{2}y_1^2 - 2x_1^2, \qquad \dot{y}_1 = -x_1 - x_1y_1, \tag{19}$$

for which an analytic change of coordinates exists in the form

$$u = \frac{x_1}{(1 + y_1)^2}, \qquad v = \frac{2y_1 + y_1^2}{2(1 + y_1)^2}, \tag{20}$$

under which system (19) becomes a linear system,

$$\dot{u} = v, \qquad \dot{v} = -u. \tag{21}$$

When $B_2 = -\frac{1}{2}$, (17) becomes

$$\dot{x}_1 = y_1 + \frac{1}{2}y_1^2 - \frac{1}{2}x_1^2, \qquad \dot{y}_1 = -x_1 - x_1y_1. \tag{22}$$

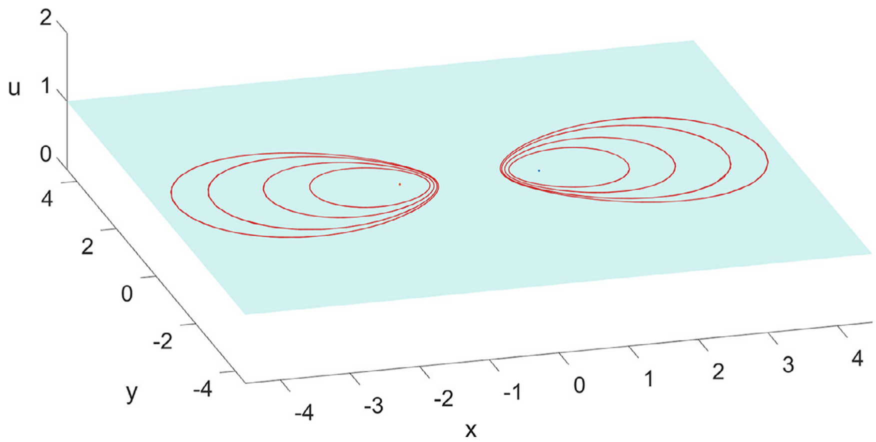

Fig. 1. Phase portrait of (9) for $\delta = 0,\ C_2 = 0,\ B_1 = 1,\ B_2 = -1,\ C_3 = 1$ and $C_1 = 1$ when the initial points chosen from the invariant surface $u = 1$.

With the transformation,

$$z = x_1 + iy_1, \qquad w = x_1 - iy_1, \qquad i = \sqrt{-1}, \tag{23}$$

system (22) is transformed to the complex system,

$$\dot{z} = -\frac{1}{2} z(z + 2i), \qquad \dot{w} = \frac{1}{2} w(-w + 2i). \tag{24}$$

Further, one can use the following analytic change of coordinates

$$Z = \frac{2z}{2 - iz}, \qquad W = \frac{2w}{2 + iw}, \tag{25}$$

to transforms system (24) to $\dot{Z} = -iZ,\ \dot{W} = iW$.

The above discussions clearly show that when $\delta = C_2 = 0$, and $B_2 = -2$ or $B_2 = -\frac{1}{2}$, the origin of system (10) is an isochronous center. □

6. Numerical simulation

Finally, in this section we present simulations to illustrate the theoretical results we have obtained in the previous sections. We first simulate the case of centers and then present a method to determine bifurcation of multiple limit cycles, and apply it to find the parameter values for simulation of the twelve limit cycles.

6.1. Simulation of centers

We use the dimensionless system (9) to demonstrate the two symmetric centers at (1,0,1) and $(-1, 0, 1)$, under the conditions:

$$\delta = 0 = C_2 = 0, \quad B_1 = C_1 = C_3 = 1, \quad B_2 = -1.$$

These two centers are located on the invariant surface, $u = 1$. Note that if one wants to use the original system (2) for the simulation, one can simply select $a_1 = b_3 = 1$ and $b_i = B_i,\ i = 1, 2,\ c_j = C_j\ j = 1, 2, 3$. All trajectories starting from the initial points on the invariant surface will remain on the surface. For example, when the initial points are chosen as

$$\begin{aligned}(x, y, u) = \ &(0.5, 0.2, 1), \quad (-0.5, 0.2, 1), \quad (0.7, 0.35, 1), \quad (-0.7, 0.35, 1),\\ &(0.6, 0.3, 1), \quad (-0.6, 0.3, 1), \quad (0.52, 0.2, 1), \quad (-0.52, 0.2, 1),\end{aligned}$$

they are all periodic orbits located on the invariant surface, as shown in Fig. 1.

However, if the initial points are chosen from outside the invariant surface, then all trajectories first converge to the invariant surface $u = 1$, and once they reach the invariant surface, they become periodic orbits on the invariant surface, as shown in Fig. 2, where four initial points are chosen above the invariant surface, given by

$$(x, y, u) = (14, 3, 4), \quad (-14, 3, 4), \quad (14, 2, 3), \quad (-14, 2, 3),$$

and another four initial points are chosen below the invariant surface, given by

$$(x, y, u) = (15, 1, -0.1), \quad (-15, 1, -0.1), \quad (15, 1, -0.5), \quad (-15, 1, -0.5).$$

They are depicted in Fig. 2(a) and (b), respectively.

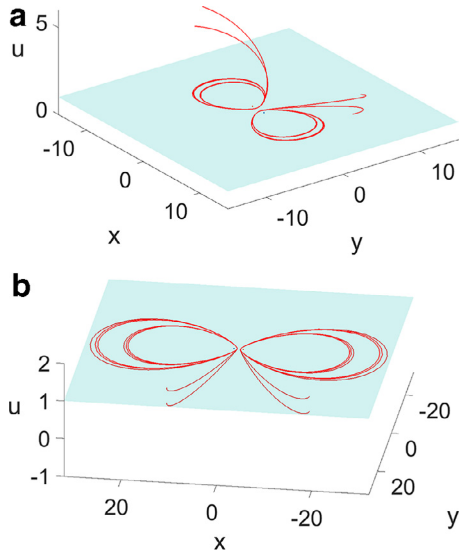

Fig. 2. Phase portraits of (9) for $\delta = 0$, $C_2 = 0$, $B_1 = 1$, $B_2 = -1$, $C_3 = 1$ and $C_1 = 1$ when the initial points chosen from outside the invariant surface $u = 1$: (a) above the surface $u = 1$; and (b) below the surface $u = 1$.

6.2. Simulation of 12 limit cycles

Next, we present simulations showing the existence of the 12 limit cycles due to Hopf bifurcation by perturbing the symmetric centers $(1,0,1)$ and $(-1, 0, 1)$ of system (9) on the invariant surface, $u = 1$.

Simulating one or two limit cycles is straightforward. However, simulating three limit cycles is not easy [32]. It is extremely hard to simulate six limit cycles around a singular point. We have two main difficulties for choosing perturbations on the parameters about the critical point. First one is that one has to find 6 positive real roots from the trundled normal form. Second one is that the multivariate polynomial equations obtained from the focus values are nonlinearly coupled.

In the following, a method is developed for finding the 12 limit cycles. We are motivated by the idea of Liu [33] and develop a more systematic approach in realizing bifurcation of multiple limit cycles and then apply it to determine the parameter values for simulating the 12 limit cycles. Suppose that the normal form of system (9) is described by polar coordinates as

$$\begin{aligned}\frac{dr}{dt} &= r(v_0 + v_1 r^2 + v_2 r^4 + v_3 r^6 + v_4 r^8 + v_5 r^{10} + v_6 r^{12} + \cdots),\\ \frac{d\theta}{dt} &= 1 + \tau_0 + \tau_1 r^2 + \tau_2 r^4 + \tau_3 r^6 + \tau_4 r^8 + \tau_5 r^{10} + \cdots,\end{aligned} \tag{26}$$

where v_k is the kth-order focus value. The first equation of (26) can be rewritten as

$$\frac{dR}{dt} = 2R(v_0 + v_1 R + v_2 R^2 + v_3 R^3 + v_4 R^4 + v_5 R^5 + v_6 R^6 + \cdots), \tag{27}$$

where $R = r^2$.

Now, the problem of solving for r^2 from $\frac{dr}{dt} = 0$ of (26) has been transformed to solving for positive real R from $\frac{dR}{dt} = 0$ of (27). Because we only focus on small-amplitude limit cycle solutions, we can introduce the scaling $R \to \epsilon R (0 < \epsilon \ll 1)$ into (27) to obtain

$$\frac{d(\epsilon R)}{dt} = 2\epsilon R(v_0 + v_1 \epsilon R + v_2 \epsilon^2 R^2 + v_3 \epsilon^3 R^3 + v_4 \epsilon^4 R^4 + v_5 \epsilon^5 R^5 + v_6 \epsilon^6 R^6 + \cdots). \tag{28}$$

Then, assume that the perturbed focus values are given as

$$v_i = K_i \epsilon^{6-i}, \quad i = 0, 1, \ldots, 5 \quad \text{and} \quad v_j = K_j + o(1), \quad j \geq 6, \tag{29}$$

under which Eq. (28) becomes

$$\frac{d(\epsilon R)}{dt} = 2\epsilon^7 R[(K_0 + K_1 R + K_2 R^2 + K_3 R^3 + K_4 R^4 + K_5 R^5 + K_6 R^6) + \epsilon R G(\epsilon, R)], \tag{30}$$

where $G(\epsilon, R)$ is analytic at (0,0).

Consider that ϵ is small enough, we know that if the equation

$$K_0 + K_1x + K_2x^2 + K_3x^3 + K_4x^4 + K_5x^5 + K_6x^6 = 0 \tag{31}$$

has six positive real roots R_1, R_2, R_3, R_4, R_5, R_6, then the equation $\frac{d(\epsilon R)}{dt} = 0$ in (30) has six positive real roots which are sufficiently close to R_1, R_2, R_3, R_4, R_5, R_6, and so system (9) has twelve limit cycles near the circles $(x-1)^2 + y^2 = \epsilon R_i$ and $(x+1)^2 + y^2 = \epsilon R_i$ on the center manifold for $i = 1, 2, \ldots 6$.

Theorem 6. *For system (9), the following perturbed parameters:*

$$\begin{aligned}
\delta &= (0.4827299955\cdots)\epsilon^6,\\
B_1 &= -(3.8845584772\cdots)\epsilon^5,\\
B_2 &= -(60.7165243976\cdots)\epsilon^4 - (274.0882036725\cdots)\epsilon^3 + (35.0371431426\cdots)\epsilon^2\\
&\quad + (1.8288009205\cdots)\epsilon - 0.1044656162\cdots,\\
C_1 &= (223.2979972269\cdots)\epsilon^3 - (28.1460377074\cdots)\epsilon^2 - (1.5888197125\cdots)\epsilon + 0.1117390817\cdots,\\
C_2 &= (43.2382890206\cdots)\epsilon^2 + (1.94126270006\cdots)\epsilon - 0.3366772533\cdots,\\
C_3 &= (3.03641314901\cdots)\epsilon + 2.9729262150\cdots,
\end{aligned} \tag{32}$$

are obtained such that the equation $\frac{d(\epsilon R)}{dt} = 0$ in (30) has six positive real roots which are sufficiently close to 1,2,3,4,5,6, and thus correspondingly, system (9) has 12 limit cycles near the circles $(x-1)^2 + y^2 = j\epsilon$ and $(x+1)^2 + y^2 = j\epsilon$ on the center manifold for $j = 1, 2, \ldots 6$.

Proof. Assume that the six positive roots of Eq. (31) are $x = 1, 2, 3, 4, 5, 6$. Then, we can use (31) to find K_i as

$$\begin{aligned}
K_0 &= 0.2413649977\cdots, \quad K_1 = -0.5913442445\cdots, \quad K_2 = 0.5444121616\cdots,\\
K_3 &= -0.2463934352\cdots, \quad K_4 = 0.05866510362\cdots, \quad K_5 = -0.0070398124\cdots,\\
K_6 &= 0.0003352291\cdots.
\end{aligned} \tag{33}$$

Clearly, δ, B_1 and B_2 can be, respectively, linearly solved from the first three focus values v_0, v_1, and v_2. Thus, we first need to determine the perturbed parameters C_1, C_2 and C_3 from the equations $v_3 = v_4 = v_5 = 0$, and then use these values to find proper perturbations on B_2 for v_2, B_1 for v_1 and δ for v_0. Then, without loss of generality, we set

$$\begin{aligned}
C_1 &= C_{1c} + k_{11}\epsilon + k_{12}\epsilon^2 + k_{13}\epsilon^3,\\
C_2 &= C_{2c} + k_{21}\epsilon + k_{22}\epsilon^2 + k_{23}\epsilon^3,\\
C_3 &= C_{3c} + k_{31}\epsilon + k_{32}\epsilon^2 + k_{33}\epsilon^3.
\end{aligned} \tag{34}$$

where C_{1c}, C_{2c} and C_{3c} are critical values such that $v_3 = v_4 = v_5 = 0$. We substitute (34) into the expressions of v_3, v_4 and v_5, and then expand them in Taylor series up to ϵ^3-order to obtain

$$\begin{aligned}
v_3 &= E_{30} + E_{31}\epsilon + E_{32}\epsilon^2 + E_{33}\epsilon^3 + o(\epsilon^3),\\
v_4 &= E_{40} + E_{41}\epsilon + E_{42}\epsilon^2 + o(\epsilon^2),\\
v_5 &= E_{50} + E_{51}\epsilon + o(\epsilon),
\end{aligned} \tag{35}$$

where E_{ij} are functions of C_1, C_2 and C_3, and so the functions of k_{ij}, $i, j = 1, 2, 3$.

Based on (29) and (33), balancing the coefficients of like powers of ϵ results in

$$\begin{aligned}
E_{30}(C_1, C_2, C_3) &= E_{31}(C_1, C_2, C_3) = E_{32}(C_1, C_2, C_3) = 0,\\
E_{40}(C_1, C_2, C_3) &= E_{41}(C_1, C_2, C_3) = E_{50}(C_1, C_2, C_3) = 0,\\
E_{33}(C_1, C_2, C_3) &= K_3 = -0.2463934352\cdots,\\
E_{42}(C_1, C_2, C_3) &= K_4 = 0.05866510362\cdots,\\
E_{51}(C_1, C_2, C_3) &= K_5 = -0.007039812435\cdots.
\end{aligned} \tag{36}$$

Then, solving the above equations gives the solutions:

$$\begin{aligned}
k_{31} &= 3.0364131490\cdots, \quad k_{22} = 43.23828902\cdots, \quad k_{13} = 223.29799722\cdots,\\
k_{11} &= -1.5888197125\cdots, \quad k_{21} = 1.94126270006\cdots, \quad k_{12} = -28.1460377074\cdots,\\
k_{23} &= k_{33} = k_{32} = 0.
\end{aligned} \tag{37}$$

So the perturbed values of the parameters C_1, C_2, and C_3 are obtained from (34) as

$$\begin{aligned}
C_1 &= (223.2979972269\cdots)\epsilon^3 - (28.1460377074\cdots)\epsilon^2 - (1.5888197125\cdots)\epsilon + 0.1117390817\cdots,\\
C_2 &= (43.2382890206\cdots)\epsilon^2 + (1.94126270006\cdots)\epsilon - 0.3366772533\cdots,\\
C_3 &= (3.03641314901\cdots)\epsilon + 2.9729262150\cdots.
\end{aligned} \tag{38}$$

Next, we assume that

$$B_2 = k_{40} + k_{41}\epsilon + k_{42}\epsilon^2 + k_{43}\epsilon^3 + k_{44}\epsilon^4. \tag{39}$$

Substituting (38) and (39) into v_2, and expanding it in Taylor series up to ϵ^4-order yield that

$$v_2 = E_{20} + E_{21}\epsilon + E_{22}\epsilon^2 + E_{23}\epsilon^3 + E_{24}\epsilon^4 + o(\epsilon^4), \tag{40}$$

where E_{ij} are functions of B_2 and so functions of k_{4i}, $i = 1, 2, 3, 4$. Combining (29) and (33), and balancing the coefficients of like powers of ϵ result in

$$E_{20} = E_{21} = E_{22} = E_{23} = 0, \;\; E_{24} = K_2 = 0.5444121616\cdots. \tag{41}$$

Then, we solve the equations in (41) to obtain the solutions,

$$\begin{aligned} k_{40} &= -0.1044656162\cdots, \quad k_{41} = 1.8288009205, \quad k_{42} = 35.0371431426\cdots, \\ k_{43} &= -274.0882036725\cdots, \quad k_{44} = -60.7165243976\cdots, \end{aligned} \tag{42}$$

for which B_2 is given by

$$\begin{aligned} B_2 = & -(60.7165243976\cdots)\epsilon^4 - (274.0882036725\cdots)\epsilon^3 \\ & + (35.0371431426\cdots)\epsilon^2 + (1.8288009205\cdots)\epsilon - 0.1044656162\cdots. \end{aligned} \tag{43}$$

Similarly, for B_1, we suppose that

$$B_1 = k_{50} + k_{51}\epsilon + k_{52}\epsilon^2 + k_{53}\epsilon^3 + k_{54}\epsilon^4 + k_{55}\epsilon. \tag{44}$$

Then, substituting (38), (43) and (44) into v_1 and expanding it in Taylor series up to ϵ^5-order results in

$$v_1 = E_{10} + E_{11}\epsilon + E_{12}\epsilon^2 + E_{13}\epsilon^3 + E_{14}\epsilon^4 + E_{15}\epsilon^5 + o(\epsilon^5). \tag{45}$$

Combining (29) and (33), and balancing the coefficients of like powers of ϵ, we have

$$E_{10} = E_{11} = E_{12} = E_{13} = E_{14} = 0, \;\; E_{15} = K_1 = -0.5913442445\cdots, \tag{46}$$

where E_{ij} are functions of B_1 and so functions of k_{5i}, $i = 1, 2, \ldots, 5$. Solving the equations in (46), we obtain the solutions,

$$k_{50} = k_{51} = k_{52} = k_{53} = k_{54} = 0, \;\; k_{55} = -3.8845584772\cdots, \tag{47}$$

and B_1 is then given by

$$B_1 = -(3.8845584772\cdots)\epsilon^5. \tag{48}$$

Finally, solving the equation $v_0 = K_0\epsilon^6 = (0.2413649977\cdots)\epsilon^6$, we obtain the value of the perturbed parameter δ as $\delta = (0.4827299955\cdots)\epsilon^6$.

Now, we have obtained all the perturbed parameter values given in (32), thus the perturbed focus values are:

$$\begin{aligned} v_0 &= (0.2413649977\cdots)\epsilon^6 + o(\epsilon^6), \quad v_1 = -(0.5913442445\cdots)\epsilon^5 + o(\epsilon^5), \\ v_2 &= (0.5444121616\cdots)\epsilon^4 + o(\epsilon^4), \quad v_3 = -(0.2463934352\cdots)\epsilon^3 + o(\epsilon^3), \\ v_4 &= (0.05866510362\cdots)\epsilon^2 + o(\epsilon^2), \quad v_5 = -(0.007039812435\cdots)\epsilon + o(\epsilon), \\ v_6 &= (0.0003352291635\cdots) + o(1), \quad \cdots. \end{aligned} \tag{49}$$

Based on the above perturbed focus values, the equation $\frac{d(\epsilon R)}{dt} = 0$ in (30) has 6 positive real roots which are sufficiently close to 1,2,3,4,5,6. Therefore, system (9) has 12 limit cycles near the circles $(x-1)^2 + y^2 = j\epsilon$ and $(x+1)^2 + y^2 = j\epsilon$ on the center manifold for $j = 1, 2, \ldots, 6$.

The proof is complete. □

For simulation, we choose $\epsilon = 2.5 \times 10^{-7}$, for which 6 positive real roots are obtained from $\frac{dr}{dt} = 0$ as follows:

$$r_1 \approx 0.0005, \quad r_2 \approx 0.0007, \quad r_3 \approx 0.0009, \quad r_4 \approx 0.0010, \quad r_5 \approx 0.0011, \quad r_6 \approx 0.0012,$$

which are the approximations of the amplitudes for the 6 limit cycles bifurcating from the center (1,0,1), and other 6 limit cycles around the center $(-1, 0, 1)$ due to symmetry. The simulation for the 6 limit cycles around the original center $(1, 0, -1)$ is shown in Fig. 3(a), with a zoomed area in Fig. 3(b). The 3 stable and 3 unstable limit cycles are shown in red and blue colors, respectively. Since $v_6 > 0$, the outer limit cycle is unstable, then the next one close to the outer one is stable, and the next one is unstable, and so on, and all the 6 limit cycles enclose the unstable focus which is near the original center $(1, 0, -1)$. It can be seen from Fig. 3 that the simulated limit cycles agree well with the analytical prediction, except for the largest one which has slightly discrepancy. The prediction indicates that all the 6 limit cycles are quite close, while the simulation shows that 5 of the 6 limit cycles are quite close but the outer one is slightly big than the predicted value.

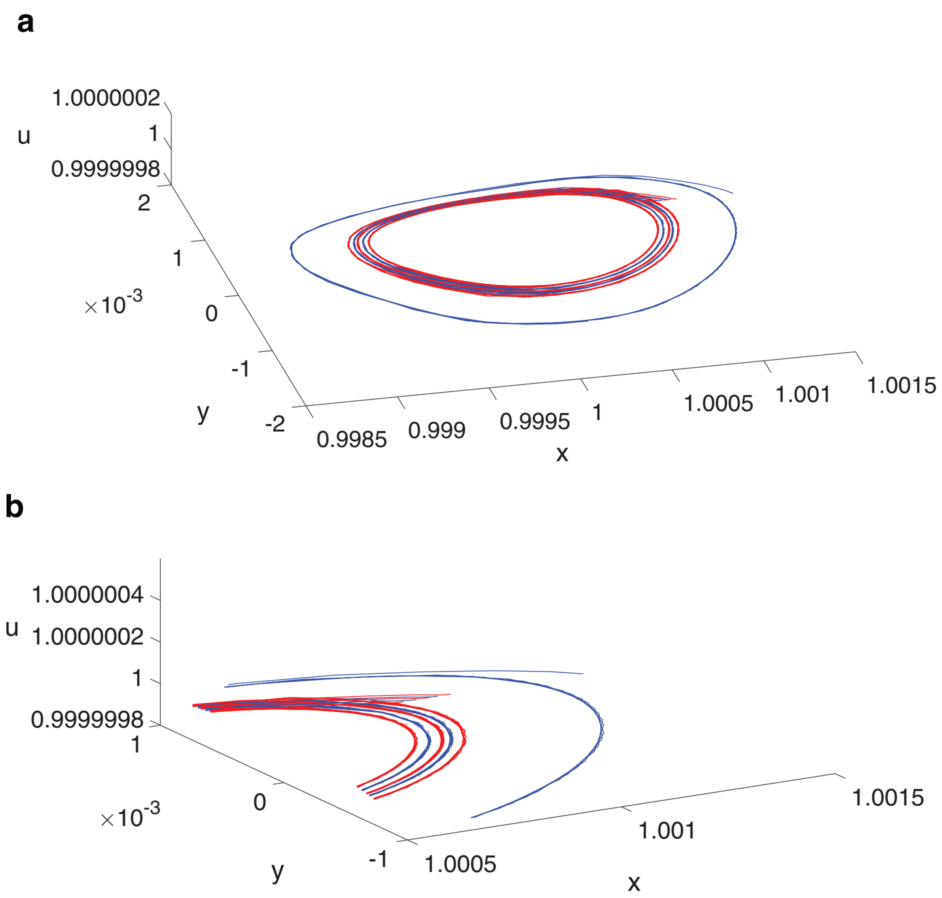

Fig. 3. Simulated six limit cycles of system (9) around (1,0,1): (a) in the $x-y-u$ space; and (b) a zoomed area of part (a).

7. Conclusion

In this paper, we have shown that at least 12 small-amplitude limit cycles exist in 3-dimensional quadratic polynomial vector fields around two symmetric critical points. The method of normal forms has been applied to compute the focus values, and then the maximal number of bifurcating limit cycles near the critical points is determined. Moreover, based on the normal forms, a set of center conditions and isochronous center conditions for the two critical points have been obtained for such 3-dimensional quadratic polynomial systems.

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